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Notes on matrix inverse over min-plus algebra

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Abstract: One of the semiring structures is the max-plus algebra, a set with entries $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$ equipped with the operation $\oplus$, which represents the maximum value, and the operation $\otimes$, which means addition. Another semiring structure is the min-plus algebra, a set with entries $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{+\infty\}$ equipped with the operation $\oplus$, representing the minimum value, and the operation $\otimes$, which means addition. Matrices over min-plus algebras can have inverses determined by certain conditions. The general inverse type can define the inverse of matrices over min-plus algebras. In this paper, we will develop the characteristics of general inverse matrices over min-plus algebras. The research method used is the literature study method sourced from books and journal articles. The main result of this study is that the generalized inverse of the matrix $A \in R_{min}^{n\times n}$ can be obtained by determining the matrix X with entry $X_{kl} = \underset{i=1}{\overset{n}{min}} \underset{j=1}{\overset{n}{min}} \left(-a_{ik} + a_{ij} - a_{lj}\right)$ which satisfies $A \otimes X \otimes A = A$.
Keywords: *Generalized, Inverse, Min-plus algebra.*

1. Introduction

Semiring is an algebraic structure obtained from rings with the condition that a ring is weakened by eliminating several ring conditions. Other algebraic structures known as Semigroups and Semirings will emerge if some properties of Groups and Rings are weakened. This shows that the algebraic structures created are Semigroups and then Semirings if some Group or Ring conditions are removed ([1], [2]). The main difference between semiring and ring structures can be seen from the existence of an inverse element for the addition operation ([3], [4]).

A structure (S,+,×) with S, a non-empty set, + addition operation, and × multiplication operation, is semiring if it fulfils the commutative and associative properties of addition, multiplication associativity, distributive, has a zero element, and a unit element ([4], [5]). It is well known that a semigroup is formed by a non-empty set S and the associative binary operation ×. Thus, the structure (S,+,×) is said to be semiring if (S,+) is a commutative semigroup, (S,×) is a semigroup, distributive, has element 0 and has a unit element ([6], [7]). With semiring entries, a semiring matrix can be developed [1], [5], [8].

One structure that is a semiring is max-plus algebra. A structure $(\mathbb{R}_{\max}, \oplus, \otimes)$ with $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ is said to be a max-plus algebra with a maximum $\oplus$ operation and an addition $\otimes$ operation. In another section, a semiring other than max-plus algebra is min-plus algebra. Min-plus algebra $\mathrm{R}_{\min} = \mathrm{R} \cup \{+\infty\}$ with minimum $(\oplus')$ and addition $(\otimes)$ operations with identity elements with respect to $\oplus'$ are $\varepsilon' = +\infty$ and $e = 0$. Max-plus algebra and min-plus algebra are isomorphic because of their similar structure. It is possible to convert the idea of max-plus algebra into min-plus algebra [5]. The semiring element has an inverse to the addition operation so that the determinant of a matrix over the semiring can be defined ([3], [6]). The inverse of the semiring matrix can be determined by determining the determinant of the semiring matrix. Looking at the characteristics of the semiring, we will specifically look at the characteristics of min-plus algebra. It is done because not all semiring properties also apply to min-plus algebra.

As with Group and Ring structures, the commutative characteristic applies to certain Semirings [6], [9], [10]. A particular Semiring owns the existence of an inverse element for addition on a Semiring. In

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a semiring, the entry on the semiring has the inverse of the + operation so that the determinant of the matrix on the semiring can be defined. This study aims to develop the characteristics of the generalized inverse matrix over a min-plus algebra.

2. Materials and Methods

Reducing several properties will form a new algebraic structure. Not all properties of the complete structure will also be reduced to the new algebraic structure. The research uses a literature study method sourced from books and journal articles. The steps for developing ideas in this study are shown in Figure 1.

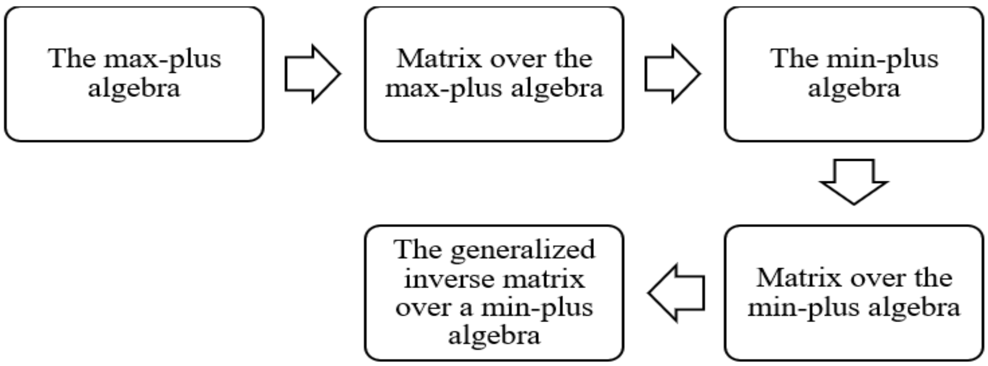

Figure 1.
Procedure for the characteristic study of min-plus algebra.

2.1. Min-Plus Algebra

The properties of a max-plus algebra can be used to create a min-plus algebra.

Definition 1
The structure $(\mathbb{R}_{min}, \oplus', \otimes)$ is said to be a min-plus algebra with $\mathbb{R}_{min} = \mathbb{R} \cup \{+\infty\}$, the binary operations $\oplus'$ and $\otimes$ defined as $a \oplus' b = min\{a, b\}$ and $a \otimes b = a + b$ for $a, b \in \mathbb{R}_{min}$. Theorem 1 provides the algebraic property of min-plus.

Theorem 1
For an $x, y, z \in \mathbb{R}_{min}$ with $e \coloneqq 0$ and $\varepsilon' = +\infty$ applies
1. Associative, namely $\forall x, y, z \in \mathbb{R}_{min}, x \oplus' (y \oplus' z) = (x \oplus' y) \oplus' z$ and $x \otimes (y \otimes z) = (x \otimes y) \otimes z$
2. Commutative, namely $\forall x, y \in \mathbb{R}_{min}, x \oplus' y = y \oplus' x$ and $x \otimes y = y \otimes x$
3. Distributive of $\otimes$ over $\oplus'$, namely $\forall x, y, x \in \mathbb{R}_{min}, x \otimes (y \oplus' z) = (x \otimes y) \oplus' (x \otimes z)$
4. There is a zero element, namely $\forall x \in \mathbb{R}_{min}, x \oplus' \varepsilon = \varepsilon \oplus' x = x$
5. There is a unit element, namely $\forall x \in \mathbb{R}_{min}, x \otimes e = e \otimes x = x$
6. There is an absorption property by the zero element ε' towards $\otimes$,that is $\forall x \in \mathbb{R}_{min}, x \otimes \varepsilon' = \varepsilon' \otimes x = \varepsilon'$
7. The idempotent property of $\oplus'$, namely $\forall x \in \mathbb{R}_{min}, x \oplus' x = x$

Meanwhile, defining the determinant uses permutation characteristics. The definition of permutation is given as follows.

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Definition 2

A permutation matrix is a matrix with exactly one entry (e) and another entry (ε') in each of its i −th row and j −th column. The permutation matrix over the min-plus algebra can be described as $P_\sigma = [p_{ij}]$ with

$$p_{ij} = \begin{cases} e; i = \sigma(j) \\ \varepsilon'; i \neq \sigma(j) \end{cases}$$

if $\sigma: \{1, 2, \dots, n\} \longrightarrow \{1, 2, \dots, n\}$ is a permutation. Thus, e appears in the $\sigma(j - th)$ row in the j −th column of P_σ.

Example 1

Let $\sigma: \{1, 2\} \rightarrow \{1, 2\}$ with $\sigma(1) = 2$ and $\sigma(2) = 1$, then

$$p_{11} = \begin{cases} e \; jika \; 1 = \sigma(1) \\ \varepsilon' \; jika \; 1 \neq \sigma(1) \end{cases}, p_{11} = \varepsilon'$$

$$p_{12} = \begin{cases} e \; jika \; 1 = \sigma(2) \\ \varepsilon' \; jika \; 1 \neq \sigma(2) \end{cases}, p_{12} = e$$

$$p_{21} = \begin{cases} e \; jika \; 2 = \sigma(1) \\ \varepsilon' \; jika \; 2 \neq \sigma(1) \end{cases}, p_{21} = e$$

$$p_{21} = \begin{cases} e \; jika \; 2 = \sigma(2) \\ \varepsilon' \; jika \; 2 \neq \sigma(2) \end{cases}, p_{22} = \varepsilon'$$

The permutation matrix is $\begin{bmatrix} \varepsilon' & e \\ e & \varepsilon' \end{bmatrix}$.

Example 2

Let $A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 5 & 8 \\ 4 & 6 & -1 \end{bmatrix}, P_\sigma = \begin{bmatrix} \varepsilon' & e & \varepsilon' \\ \varepsilon' & \varepsilon' & e \\ e & \varepsilon' & \varepsilon' \end{bmatrix}$ we have

$$A \otimes P_\sigma = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 5 & 8 \\ 4 & 6 & -1 \end{bmatrix} \otimes \begin{bmatrix} \varepsilon' & e & \varepsilon' \\ \varepsilon' & \varepsilon' & e \\ e & \varepsilon' & \varepsilon' \end{bmatrix} = \begin{bmatrix} -2 & 1 & 3 \\ 8 & 3 & 5 \\ -1 & 4 & 6 \end{bmatrix}$$

The right-hand multiplication of P_σ creates a permutation of the matrix columns so that the i −th column of A appears as the $\sigma(i)$ −th column of $A \otimes P_\sigma$. The permutation matrix P_σ has an inverse, namely $P_{\sigma^{-1}}$ where $P_{\sigma^{-1}}$ is the transpose of P_σ obtained $P_{\sigma^{-1}} = P_{\sigma^T}$ so that $P_\sigma \otimes P_{\sigma^{-1}} = E$.

If A is a matrix over a field, then a single matrix B must satisfy the property $A \otimes B \otimes A = A$. A matrix B that satisfies this property is called the generalized inverse of matrix A. In min-plus algebra, there is no guarantee that every matrix has a generalized inverse. If A has a generalized inverse, then A is considered regular. We will discuss determining whether a matrix A is a regular min-plus algebra.

A min-plus algebra can be formed based on the characteristics of a max-plus algebra. As an initial characteristic, the following theorem is given.

Theorem 2

Given an idempotent commutative semigroup $(S, +)$. If on S a relation $\geq$ is defined by $b \geq a \Leftrightarrow a + b = b$, then the relation $\leq$ is a partial order on S.

Proof :

Given any $a, b, c \in S$ then

1. Since S is idempotent, then $a + a = a \Leftrightarrow a \geq a$

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2. If $b \geq a$ and $a \geq b$, then $a + b = b$ and $b + a = a$. Since S is commutative, then $a = b$
3. If $b > a$ and $a > c$ then $a + b = b$ and $c + a = a$ then
$$b + c = (a + b) + c = (b + a) + c = b + (a + c) = b + a = b$$
So, we have $b > c$.

Definition 3
In $\mathbb{R}_{min}$, the relation $\geq_{min}$ is defined as
$x \geq_{min} y \iff x \oplus' y = y$
Theorem 3
The relation $\geq_{min}$ is a partial order.
Proof:
Given $a, b, c \in \mathbb{R}_{min}$, then
1. Since $\mathbb{R}_{min}$ is idempotent then $a \oplus' a = \min\{a, a\} = a$ therefore $a \geq_{min} a$
2. If $a \geq_{min} b$ and $b \geq_{min} a$ then $a \oplus' b = b$ and $b \oplus' a = a$. Since $\mathbb{R}_{min}$ is commutative then $a = b$
3. If $a \geq_{min} b$ and $b \geq_{min} c$ then $a \oplus' b = b$ and $b \oplus' c = c$ then
$$a \oplus' c = a \oplus' (b \oplus' c) = a \oplus' (b \oplus' c) = (a \oplus' b) \oplus' c = b \oplus' c = c$$
So, $a \geq_{min} c$.

Definition 4
For $\mathbb{R}_{min}$, we use the parsial ordered $\geq_{min}$, that is $a \geq_{min} b \iff a \oplus' b = \min(a, b) = b$. The structure $(\mathbb{R}_{min}, \leq)$ is a partially ordered set (poset).

Theorem 4
Let $A \in M_n(\mathbb{R}_{min})$ and supposed $L_A: \mathbb{R}^n_{min} \longrightarrow \mathbb{R}^n_{min}$ with $L_A(x) = A \otimes x$. We have $A = P_\sigma \otimes D(\lambda_i)$ for a permutation and $\lambda_i > \varepsilon$ if and only if L_A injective.
Proof :
$(\Longrightarrow)$ $L_A(x) = L_A(x')$ such as $A \otimes x = A \otimes x'$ so $x = x'$.
$(\Longleftarrow)$ It is known that L_A is injective. For each i can be defined $F_i = \{j | a_{ji} > \varepsilon\}$ and $G_i = \{j | a_{jk} > \varepsilon, k \neq i\}$. We called $F_i \subseteq G_i$; the contradiction assumes that $F_i \subseteq G_i$. We will show a contradiction with injective L_A.
Let $x = [x_k]$ with $x_k = \begin{cases} e\,; k \neq i \\ \varepsilon\,; k = i \end{cases}$.
Suppose $b = A \otimes x = \bigotimes_{k \neq i} a_{*k}$ with a_{*k} defined the $k-$th column of A.
Suppose $j \in F_i$, then $j \in G_i$. Its mean that $k \neq i$ for $a_{jk} > \varepsilon$.
In $\mathbb{R}_{min}$, we can complete the order relation $\leq$, namely $a \leq b$ if and only if $a \oplus' b = a$.
So $(\mathbb{R}_{min}, \leq)$ is a poset (partially ordered set).

Definition 5
A mapping f on a partially ordered set is said to be isotone if for $x \leq y$ the result is $f(x) \leq f(y)$.

Example 3
Given $f: \mathbb{R}_{min} \longrightarrow \mathbb{R}_{min}$ with $f(x) = x \otimes' 7$ is an isotone mapping, namely for every $x \leq y$ results in $x - 7 \leq y - 7$ results in $f(x) \leq f(y)$.

Definition 6
Given $(E, \leq)$ is a poset and $A \subseteq E$.

i) For $a \in A$ there is $x \in A$ resulting in $a \leq x$ so a is called minimum
ii) For $a \in A$ it is called the minimal element of A if there is $x \in A$ with $a \leq x$ then $a = x$.

Definition 7

An isotone mapping $f: D \longrightarrow E$ with D, E poset is said to be a residual mapping if for all $b \in E$ then $\{x|b \leq f(x)\}$ has a minimum element denoted $f^{\#}(b)$. The isotone mapping $f^{\#}: E \longrightarrow D$ is called residual of f.

Theorem 5

If $f: D \rightarrow E$ is a residualized mapping, then the equation $f(x) = b$ has a solution if and only if $f(f^{\#}(b)) = b$.

Proof:

(⇒) Given $f(x) = b$ has a solution, say x_1. We get $f(x_1) = b$. Since $f^{\#}(b)$ is a minimal element in $\{x|f(x) \leq b\}$, then $x_1 \leq f^{\#}(b)$. Since f is isotone then $f(x_1) \leq f(f^{\#}(b))$, according to (*) $f(f^{\#}(b)) \leq b$, consequently $b = f(x_1) \leq ff^{\#}(b) \leq b$, namely $f(f^{\#}(b)) = b$.

(⇐) Given $f(f\hat{}\#(b)) = b$, then the equation $f(x) = b$ has a solution, namely $x = f^{\#}(b)$.

The function f is residualized, because $y \in \mathbb{R}_{min}$ with $\{x|y \leq x \otimes' 7 = f(x)\}$ is a minimal element, namely $x = f^{\#}(y) = y + 7$.

Definition 8

For every $b \in E$ then $\{x|b \leq f(x)\}$ has a minimal element denoted $f^{\#}(b)$. For $y \in \mathbb{R}_{min}$ $\{x|y \leq x \otimes' 7 = f(x)\}$ the minimal element is $x = f^{\#}(y) = y + 7$.

Definition 9

A subsolution of $A \otimes x = b$ is x that satisfies $A \otimes x \geq b$, a linear system for obtaining the general result of the equation $A \otimes x = b$. An ordered pair of vectors is defined by $x \geq y$ if $x \oplus' y = y$.
Since $A \in R_{min}^{n \times n}$ and $X \in R_{min}^{n \times n}$ then

$$A \otimes x = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + x_1 \oplus' a_{12} + x_2 \oplus' \cdots \oplus' a_{1n} + x_n \\ a_{21} + x_1 \oplus' a_{22} + x_2 \oplus' \cdots \oplus' a_{2n} + x_n \\ \vdots \\ a_{n1} + x_1 \oplus' a_{n2} + x_2 \oplus' \cdots \oplus' a_{nn} + x_n \end{bmatrix}$$

$$= \begin{bmatrix} \oplus' a_{1j} + x_j \\ \oplus' a_{2j} + x_j \\ \vdots \\ \oplus' a_{nj} + x_j \end{bmatrix}, j = 1,2, \dots, n$$

with $\oplus' a_{1j} + x_j = min\{a_{11} + x_1, a_{12} + x_2, \cdots, a_{1n} + x_n\}$

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Form $f_j(x_j) = \begin{bmatrix} a_{1j} + x_1 \\ a_{2j} + x_2 \\ \vdots \\ a_{nj} + x_n \end{bmatrix}$ so that $A \otimes x = \overset{n}{\underset{j=1}{\oplus'}} f_j(x_j)$. So, $A \otimes x = f_1(x_1) \oplus' f_2(x_2) \oplus' \cdots \oplus' f_n(x_n)$.

For each j, if $x_{jh} \leq x_{jk} \Longrightarrow f_j(x_{jh}) \leq f_j(x_{jk})$. According Definition 5 and Theorem 5, so $\{x | A \otimes x \geq b\}$ has a minimum element denoted $A^{\#}(b)$.

Therefore, to determine the solution of the equation $Ax = b$, check wether $A\left(A^{\#}(b)\right) = b$. The following is a theorem that states this characteristic.

Theorem 6

If $A \in R_{min}^{n\times n}$ and $b \in R_{min}^{n}$, then the equation $A \otimes x = b$ has a solution if and only if $A\left(A^{\#}(b)\right) = b$. In other words, the solution is $x = A^{\#}(b)$.

Proof :

$(\Longrightarrow)$

It is known that the equation $A \otimes x = b$ has a solution x^*, namely $A \otimes x^* = b$, so that $A \otimes x^* \geq b$.

Since $A^{\#}b$ is the minimal element in $\{x | A \otimes x \geq b\}$, then $x^* \geq A^{\#}b$. It is obtained

$$A \otimes x^* \leq A(A^{\#}b)$$

$$\Leftrightarrow\ b = A \otimes x^* \geq A\left(A^{\#}b\right) \quad \ldots\ldots\ldots..(*)$$

Furthermore, according to Theorem 5

$$A\left(A^{\#}b\right) \geq b \quad \ldots\ldots\ldots\ldots(**)$$

From $(*)$ and $(**)$ it is obtained $A\left(A^{\#}(b)\right) = b$.

$(\Longleftarrow)$

It is known that $A\left(A^{\#}b\right) = b$. So the equation $A \otimes x = b$ has a solution, namely $x = A^{\#}b$.

Therefore, $A \otimes x = b$ has a solution $x = A^{\#}(b)$. It means that $A\left(A^{\#}(b)\right) = b$.

For example, if $A \otimes x = b$ has solution x_j, then there is a smallest subsolution $A \otimes x = b$.

$$A \otimes x \geq b \ \Leftrightarrow \underset{j}{\oplus'} A_{ij} \otimes x_j \geq b_i, \forall i \text{ then}$$

$$A_{i1} \otimes x_1 \oplus' A_{i2} \otimes x_2 \oplus' \ldots \oplus' A_{in} \otimes x_n \geq b_i$$

$$i = 1 \ \Longrightarrow A_{11} \otimes x_1 \oplus' A_{12} \otimes x_2 \oplus' \ldots \oplus' A_{1n} \otimes x_n \geq b_1$$

$$i = 2 \ \Longrightarrow A_{21} \otimes x_1 \oplus' A_{22} \otimes x_2 \oplus' \ldots \oplus' A_{2n} \otimes x_n \geq b_2$$

$$\vdots$$

$$i = n \ \Longrightarrow A_{n1} \otimes x_1 \oplus' A_{n2} \otimes x_2 \oplus' \ldots \oplus' A_{nn} \otimes x_n \geq b_n$$

for i, $\min\{A_{i1} + x_1, A_{i2} + x_2, \ldots, A_{in} + x_n\} \geq b_i$.

For i, j, we have $A_{ij} + x_j \geq b_i$, which results in $x_j \geq b_i - A_{ij}$.

Obtain $x_j \geq max\{b_i - A_{ij}\}$ for each i. Next, $-x_j \leq min\{-b_i + A_{ij}\}$ for each i.

In other words $-x_j \leq min\{-b_i \otimes A_{ij}\}$ for each i.

So that $-x_j = min\{-b_i \otimes A_{ij}\}$subsolution of $A \otimes x = b$ or expressed $-x_j = min\left\{\left(A_{ij}\right)^t \otimes (-b_j)\right\}$.

Example 4

Given a system of linear equations over min-plus algebra

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$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

We get

$$\begin{aligned} -x_j &= min\{-b_i \otimes A_{ij}\} \\ &= [-6 \quad -7] \otimes \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \\ &= [-4 \oplus' -3 \quad -3 \oplus' -2] \\ &= [-4 \quad -3] \end{aligned}$$

So, $x_j = [4 \quad 3]$. Or, $-x_j = min\left\{\left(A_{ij}\right)^t \otimes \left(-b_j\right)\right\} = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} \otimes \begin{bmatrix} -6 \\ -7 \end{bmatrix} = \begin{bmatrix} -4 \oplus' -3 \\ -3 \oplus' -2 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

It can be proven that

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \otimes \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \oplus' 6 \\ 8 \oplus' 8 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix} \geq \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

3. Results and Discussion

Based on the explanation above, the following properties are obtained.

Definition 7

A matrix $A \in M_n(R)$ is said to be invertible over the min-plus algebra if there is a matrix B such that $A \otimes B = E$ with E the identity matrix over the min-plus algebra and is denoted $B = A^{\otimes^{-1}}$.

To determine the inverse matrix over min-plus algebra, permutation is required.

Definition 8

If $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}_{min}, \lambda_i \neq \varepsilon$ then the diagonal matrix is defined as follows.

$$D(\lambda_i) = \begin{bmatrix} \lambda_1 & \varepsilon & \cdots & \varepsilon \\ \varepsilon & \lambda_2 & \vdots & \varepsilon \\ \cdots & \cdots & \ddots & \cdots \\ \varepsilon & \varepsilon & \cdots & \lambda_n \end{bmatrix}$$

Theorem 7

Given $A \in M_n(R_{\min})$. If and only if there is a permutation σ and values $\lambda_i > \varepsilon, i \in \{1,2,\dots,n\}$ such that $A = P_\sigma \otimes D(\lambda_i)$, then $A \in M_n(R_{\min})$ has a left inverse.

Proof:

($\Longrightarrow$) Given $A \in M_n\ (R_{min})$, there exist B so that it satisfies the equation $A \otimes B = E$, meaning

(1) $min_k(a_{ik} + b_{ik}) = e = 0$ for every k there is i so that $a_{ik} + b_{ki} = e$, we have the function $i = \theta(k)$ with $a_{i\theta(i)} > \varepsilon$ and $b_{\theta(i)i} > \varepsilon$.

(2) $min_k\left(a_{ik} + b_{kj}\right) = \varepsilon' = \infty$ for all $i \neq j$

Based on (2), it is obtained

(3) $a_{i\theta(j)} = \varepsilon'$ for all $i \neq j$.

Since $a_{i\theta(i)} > \varepsilon' = a_{i\theta(j)}$ for all $i \neq j$ then θ is an injection and permutation function. Meanwhile, $a_{i\theta(i)}$ is a single entry in the $\theta(i)$ −th column of A, which is not ε'. For example, $\hat{A} = P_\theta \otimes A$. The $\theta(i)$ −th row of $\hat{A}$ is the i −th row of A, which has a larger entry then ε' in the $\theta(i)$ −th column.

Thus, all larger $\hat{A}$ diagonal entries become ε'. A has only one non$-\varepsilon'$ entry in each column, which is also true for $\hat{A}$.

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So we get $P_\theta \otimes A = \hat{A} = D(\lambda_i)$ with $\lambda_i = a_{\theta^{-1}(i)i} > \varepsilon'$.
Suppose, $\sigma = \theta^{-1}$, because $P_\sigma \otimes P_\theta = P_{\theta^{-1}} \otimes P_\theta = E$, then $A = P_\sigma \otimes D(\lambda_i)$.
So, it is proven that $A = P_\sigma \otimes D(\lambda_i)$.
($\Leftarrow$) Assume $A = P_\sigma \otimes D(\lambda_i)$ with $\lambda \in R_{min}$ and $\lambda_i > \varepsilon$.
If the statement is true then for example $B = P_{\sigma^{-1}} \otimes D(-\lambda_i)$, with $-\lambda_i = \lambda_i^{\otimes^{-1}}$. So we have

$$\begin{aligned} A \otimes B &= \left(P_\sigma \otimes D(\lambda_i)\right) \otimes (P_{\sigma^{-1}} \otimes D(-\lambda_i)) \\ &= P_\sigma \otimes \left(D(\lambda_i) \otimes D(-\lambda_i)\right) \otimes P_{\sigma^{-1}} \\ &= P_\sigma \otimes E \otimes P_{\sigma^{-1}} \\ &= P_\sigma \otimes P_{\sigma^{-1}} \\ &= E \end{aligned}$$

And, $A \otimes B = E$ and B is the right inverse of A.

From the theorem above, we get the necessary and sufficient conditions for matrix A to be invertible over min-plus algebra, namely matrix A is invertible if and only if matrix A is a permuted diagonal matrix with $A = P_\sigma \otimes D(\lambda_i)$.

The purpose of finding the generalized inverse is to determine the solution of the linear equation system $AXB = C$. A matrix $A \in R_{min}^{n\times n}$ has a generalized inverse matrix $X \in R_{min}^{n\times n}$ if $A \otimes X \otimes A = A$. So, it can be said that the generalized inverse is the smallest subsolution of the equation $A \otimes X \otimes A = A$. Several steps are required to determine matrix X as the generalized inverse of equation $A \otimes X \otimes A = A$.

Definition 9

For a matrix $A \in R_{min}^{n\times n}$, then the matrix $B \in R_{min}^{n\times n}$ is said to be the generalized inverse of the matrix A if $A \otimes B \otimes A = A$ is satisfied.

To determine whether or not there is a matrix B that satisfies $A \otimes B \otimes A = A$, is equivalent to determining whether or not there is a solution to the equation $A \otimes X \otimes A = A$ with $A \in R_{min}^{n\times n}$.

1. Bring the equation $A \otimes X \otimes A = A$ to the form $Ax = b$.
2. Determine the matrix X
3. Prove that the matrix X is a generalized inverse by substituting it into the equation $A \otimes X \otimes A = A$.

Given $A \in R_{min}^{n\times n}$ with operations $\oplus'$ and $\otimes$. The ij −th element in $A \otimes X \otimes A$ with $1 \leq i, j \leq n$ in $A \otimes X \otimes A$ is

$$\begin{aligned} & [A \otimes X \otimes A]_{ij} = A_{ij} \\ \Leftrightarrow\ & [A \otimes X]_{il} \otimes A_{lj} = A_{ij} \qquad 1 \leq l \leq n \\ \Leftrightarrow\ & A_{ik} \otimes X_{kl} \otimes A_{lj} = A_{ij} \qquad 1 \leq k, l \leq n \\ \Leftrightarrow\ & A_{ik} \otimes \bigoplus_{l=1}^{n}{}' x_{kl} \otimes A_{lj} = A_{ij} \qquad 1 \leq k \leq n \\ \Leftrightarrow\ & \bigoplus_{k=1}^{n}{}' A_{ik} \otimes \bigoplus_{l=1}^{n}{}' x_{kl} \otimes A_{lj} = A_{ij} \\ \Leftrightarrow\ & \bigoplus_{k=1}^{n}{}' \bigoplus_{l=1}^{n}{}' A_{ik} \otimes x_{kl} \otimes A_{lj} = A_{ij} \end{aligned}$$

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$$\Leftrightarrow \bigoplus_{k=1}^{n}{}' \left[\bigoplus_{l=1}^{n}{}' \left(A_{ik} \otimes X_{kl} \otimes A_{lj} \right) \right] = A_{ij}$$

So we get $\bigoplus_{i=1}^{n}{}' \bigoplus_{j=1}^{n}{}' f_{ij}(X_{kl}) = A_{ij}$.

The generalized inverse can be found by solving the equation $A \otimes x = b$ in min-plus algebra. For the generalized inverse, denoted $A^{\otimes -1}$, the form $A \otimes X \otimes A = A$ is brought to the form $A \otimes x = b$. For the ij −th element, it is obtained as follows:

$$\Leftrightarrow [A \otimes X \otimes A]_{ij} = A_{ij}$$
$$\Leftrightarrow A_{ik} \otimes X_{kl} \otimes A_{lj} = A_{ij}$$
$$\Leftrightarrow A_{ik} \otimes X_{kl} \otimes A_{lj} \otimes A_{lj}^{\otimes -1} = A_{ij} \otimes A_{lj}^{\otimes -1}$$
$$\Leftrightarrow A_{ik} \otimes X_{kl} \otimes E = A_{ij} - A_{lj}$$

If suppose $b = A_{ij} - A_{lj}$ then we write

$$\Leftrightarrow A_{ik} \otimes X_{kl} = b$$
$$\Leftrightarrow E \otimes X_{kl} = -A_{ik} \otimes b$$
$$\Leftrightarrow X_{kl} = -A_{ik} \otimes b$$

$$\Leftrightarrow X_{kl} = -\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i1} & \cdots & a_{ik} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$\Leftrightarrow X_{kl} = \begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1k} \\ -a_{21} & -a_{22} & \cdots & -a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{i1} & -a_{i1} & \cdots & -a_{ik} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

For $X_{kl} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1l} \\ x_{21} & x_{22} & \cdots & x_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k1} & x_{k2} & \cdots & x_{kl} \end{bmatrix}$

The result,

$$x_{11} = \min\{-a_{11} + b_{11}, -a_{12} + b_{21}, \dots, -a_{1k} + b_{n1}\}$$
$$x_{12} = \min\{-a_{11} + b_{12}, -a_{12} + b_{22}, \dots, -a_{1k} + b_{n2}\}$$
$$\vdots$$
$$x_{1l} = \min\{-a_{11} + b_{1n}, -a_{12} + b_{2n}, \dots, -a_{1k} + b_{nn}\}$$
$$x_{21} = \min\{-a_{21} + b_{11}, -a_{22} + b_{21}, \dots, -a_{2k} + b_{n1}\}$$
$$x_{22} = \min\{-a_{21} + b_{12}, -a_{22} + b_{22}, \dots, -a_{2k} + b_{n2}\}$$
$$\vdots$$
$$x_{2l} = \min\{-a_{21} + b_{1n}, -a_{22} + b_{2n}, \dots, -a_{2k} + b_{nn}\}$$
$$\vdots$$
$$x_{k1} = \min\{-a_{i1} + b_{11}, -a_{i2} + b_{21}, \dots, -a_{ik} + b_{n1}\}$$
$$x_{k2} = \min\{-a_{i1} + b_{12}, -a_{i2} + b_{22}, \dots, -a_{ik} + b_{n2}\}$$
$$\vdots$$
$$x_{kl} = \min\{-a_{i1} + b_{1n}, -a_{i2} + b_{2n}, \dots, -a_{ik} + b_{nn}\}$$

So, it can be written as

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$$X_{kl} = \begin{bmatrix} \min\{-a_{1k} + b_{n1}\} & \min\{-a_{1k} + b_{n2}\} & \cdots & \min\{-a_{1k} + b_{nn}\} \\ \min\{-a_{2k} + b_{n1}\} & \min\{-a_{2k} + b_{n2}\} & \cdots & \min\{-a_{2k} + b_{nn}\} \\ \vdots & \vdots & \ddots & \vdots \\ \min\{-a_{ik} + b_{n1}\} & \min\{-a_{ik} + b_{n2}\} & \cdots & \min\{-a_{ik} + b_{nn}\} \end{bmatrix}$$

Suppose $b = A_{ij} - A_{lj}$, we have $X_{kl} = \min_{i=1}^{n} \min_{j=1}^{n} \left(-a_{ik} + a_{ij} - a_{lj}\right)$

From the form x_{kl} , we obtain a form that can be expressed $\bigoplus_{k=1}^{m} \left[\bigoplus_{l=1}^{m} (A_{ik} + X_{kl} + A_{lj}) \right] = A_{ij}$.
by determining whether or not there is a solution to the equation $A \otimes X \otimes A = A$ with $A \in \mathbb{R}_{min}^{n\times m}$.
The ij −th element in $A \otimes X \otimes A$ is $[A \otimes X \otimes A]_{ij} = \bigoplus'_{k=1}^{m} \bigoplus'_{l=1}^{n} \left(A_{ik} + X_{kl} + A_{lj}\right)$. So, we get the equation

$$\bigoplus'_{k=1}^{m} \left[\bigoplus'_{l=1}^{n} \left(A_{ik} + X_{kl} + A_{lj}\right) \right] = A_{ij} \quad \ldots\ldots\ldots\ldots\ldots(3)$$

If for each k, l is formed

$$f_{ij}(X_{kl}) = A_{ik} + X_{kl} + A_{lj} \quad \ldots\ldots..(4)$$

then equation (3) becomes

$$\bigoplus_{i=1}^{n} \bigoplus_{i=1}^{m} f_{ij}(X_{kl}) = A_{ij} \quad \ldots\ldots\ldots(5)$$

It is obtained that $X_{kl} = \min_{i=1}^{n} \min_{j=1}^{n} \left(-a_{ik} + a_{ij} - a_{lj}\right)$ corresponds if substituted into the equation $A \otimes X \otimes A = A$.

4. Conclusions

In this conclusion, the property is obtained that permutation is required determining the inverse matrix over min-plus algebra. We obtain several theorems that show the characteristics of the inverse of matrices in min-plus algebra, especially generalized matrices. The characterization is obtained by considering the characteristics of the solutions linear equations system over min-plus algebra. The generalized inverse of the matrix $A \in R_{min}^{n\times n}$ can be obtained by determining the matrix X with entry $X_{kl} = \min_{i=1}^{n} \min_{j=1}^{n} \left(-a_{ik} + a_{ij} - a_{lj}\right)$ which satisfies $A \otimes X \otimes A = A$.

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